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SOME NEW APPROACHES TO RISK

by

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1. Introduction

The problems involved in making investments under risky conditions have been, and possibly always will be, a challenge to persons concerned with management processes. It should therefore be no surprise to observe that management scientists and practitioners of operations research, along with others, should now be experiencing and responding to this challenge. A variety of proposed innovations for use by management has emerged and some of the more recent of these innovations will be covered as they appear to be of interest in this paper.[1]

A great deal can be (and has been) said on the subject of risk and how it might be identified, measured and evaluated. It is not the purpose of this paper to distinguish rigorously between different categories or dimensions of risk.[2] Rather we shall conceptualize "risk" as emerging from the fact that some of the information which is pertinent to a decision can at best be known only in the form of specified probability distributions. The resulting possibility of deviations from any estimate of the events governed by such probability distributions is then the basic phenomenon which we shall suppose gives rise to risk.

Of course, more than one probability distribution may be applicable and a combination of these distributions may then also require consideration

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1/ A good, relatively up-to-date survey of related topics, especially with reference to deterministic (non-risky) conditions for use in capital budgeting, may be found in H. M. Weingartner [76].

2/ E.g., in this paper we shall use the terms "risk" and "uncertainty" interchangeably and hence we will not make distinctions such as are to be found in Lutz and Lutz [43] pp. 179 ff. or Hirschleifer [38] and [39].

prior to effecting choices between investment alternatives. This kind of
phenomenon can supposedly be handled, at least in principle, by suitable
theorems or algorithms in probability and statistics. At any rate, given
this assumption, one version of a more classical approach would next proceed
to reduce each alternative to a single-number basis for comparison. In more
sophisticated analyses this might be accomplished via a "utility function"
approach.[1] Other versions proceed through discount rate adjustments in
order to obtain present-value calculations which allow for risk or
provided "bogey rates" of internal return as well as, perhaps, shortened (or
altered) payback period[2] allowances, and other such devices. In any case
these reductions are supposed to permit all investment alternatives to be
ordered on a single scale which measures their degree of attractiveness
while making due allowance for risk.

Some of the approaches we shall examine are also concerned with choices
that maximize a single figure of merit. Others are concerned with developing
the relevant combinations of probability distributions so that these may
themselves be used as a basis for managerial choice. Evidently the latter
collection of suggested approaches differs, at least by emphasis, from those
described in the preceding paragraph. The same is also true of the approaches
we shall also describe as proceeding by reference to a single figure-of-merit
optimization.[3] This difference is in emphasis only, of course, but it is

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1/   Vide, e.g., Raiffa and Schlaifer [56].

2/   In some cases, "payback plus" may be used--e.g., by altering the computations
     so that the recovery must include depreciation and possibly other additional
     elements as well as the original investment.

3/   These may also be extended to optimizations involving more than one figure
     of merit. See, e.g., [14] and [17].

nonetheless important not only as a guide for staging the analytical development but also as a way of ensuring managerial understanding of the choices to be made. Thus, in particular, the more classical approaches emphasize the desirability of ascertaining the way choices should be made at the outset so that, e.g., managerial analysts or subordinates could then be governed accordingly. The newer approaches which we shall discuss proceed in a rather different sequence and hence are likely to treat the risks and other aspects of the problem so that they can be considered in explicit detail as a part of the model which is to be employed. It is thus necessary then to consider the nature and meaning of risk and how its different dimensions might be treated via constraints or the criteria that enter into a composite figure of merit.

To avoid possible misunderstanding it should be said, at this point, that this paper is not concerned with issues such as whether a "present value" provides a better figure of merit than an "internal rate of return" via a "bogey adjustment" or a "payback period" computation.[1] Indeed it will be one purpose of this paper to suggest that some of these issues might be resolved-- or at least placed in a different perspective--if some of the new methodologies can make it possible to avoid insisting on the use of one of these figures to the exclusion of all others. The main emphasis in this paper, however, is on some of these newer methodological innovations and hence we shall be able to deal with these figure-of-merit topics and related issues only by example.[2]

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[1] This paper is also not concerned with problems such as data discovery and treatment, administration, implementation and control aspects of capital budgeting.

[2] For further discussion of this methodology in terms of its impact on such issues see, e.g., Byrne, Charnes, Cooper and Kortanek [7] or Charnes and Cooper [11].

One such set of innovations revolves around a variety of simulation approaches. This includes such topics as "Risk Analysis"[1] and "Venture Analysis."[2] Another approach involves the use of "Decision Trees" which has recently been joined to "Risk Analysis" in a set of techniques (and related concepts) which have been called "Stochastic Decision Trees."[3] The point of these approaches, as will be seen, is to develop the risk aspects of decisions by reference to the underlying probability distributions. That is, it is supposed that it may be better to bring these "distributional aspects" of the problem into prominence explicitly rather than to suppose a prior analysis in which all aspects of choice have previously been attended to by means such as discount-rate adjustments, etc.

In addition to the already indicated approaches, another set has also been evolving from recent extensions of Linear Programming which include, inter alia, Stochastic Linear Programming, Linear Programming Under Uncertainty, and Chance Constrained Programming.[4] Here, too, attention has been directed to dealing with the underlying probability distributions in all detail-- in order to decide how best to combine and choose between different probabilistic (i.e., risk) combinations--although contact is also made with previously available versions of the choice problem by means of suitably devised constraints and objectives.

The latter approaches emphasize analytic or mathematical models that are formulated with explicitly stated optimization objectives whereas, e.g.,

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1/ See D. B. Hertz [34].

2/ As in Hess and Quigley [33].

3/ Vide Hespos and Strassmann [35]. The term "decision trees"--which was adapted by them from Magee [47] and [48]--seems to have had its first appearance in Raiffa and Schlaifer [56], Chapter 1, which should be consulted in any event for discussions that relate these ideas to constructs employed in the theory of games and statistical decision theory.

4/ See references in the bibliography.

"Risk Analysis" proceeds by reference to computer simulation models in which such optimizations are only implicitly present. It is not proposed to emphasize these differences and, indeed, a very different approach will be taken and analytical models with explicit optimization objectives will be supplied in all cases in order to provide a more uniform way of relating these approaches to one another.

Much research remains to be done in the simulation as well as the optimization approaches referred to above. This might well include ways of relating these simulation and optimization (and like) approaches in order to clarify and perhaps lend added power and flexibility to the whole. Such work is going forward along with work on computer codes, analytic characterizations, development of algorithms, etc. But it is not necessary to wait until this has all been accomplished. Useful things can now be said about each of these topics, their possible relations and how they might be used separately or together in potential applications. The latter may serve in turn to sharpen some of the issues for research and so on. This, at any rate, is the purpose of this paper.

2. A New Product Example of a Stochastic Decision Tree:

To bring the points at issue into focus as quickly as possible we now turn to the example of Figure 1 which is adapted from Hespos and Strassmann [35] with the following interpretation. It is supposed that a decision has been made to introduce a new product [1] but, as yet, there has been no

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[1] This example is only illustrative and hence should not be confused with models such as DEMON--see [12] and [13]--which are (a) designed for actual application and (b) necessarily reflect considerably more sophisticated concepts and machinery than can be dealt with in any detail in the present paper.

FIGURE 1

A STOCHASTIC DECISION TREE
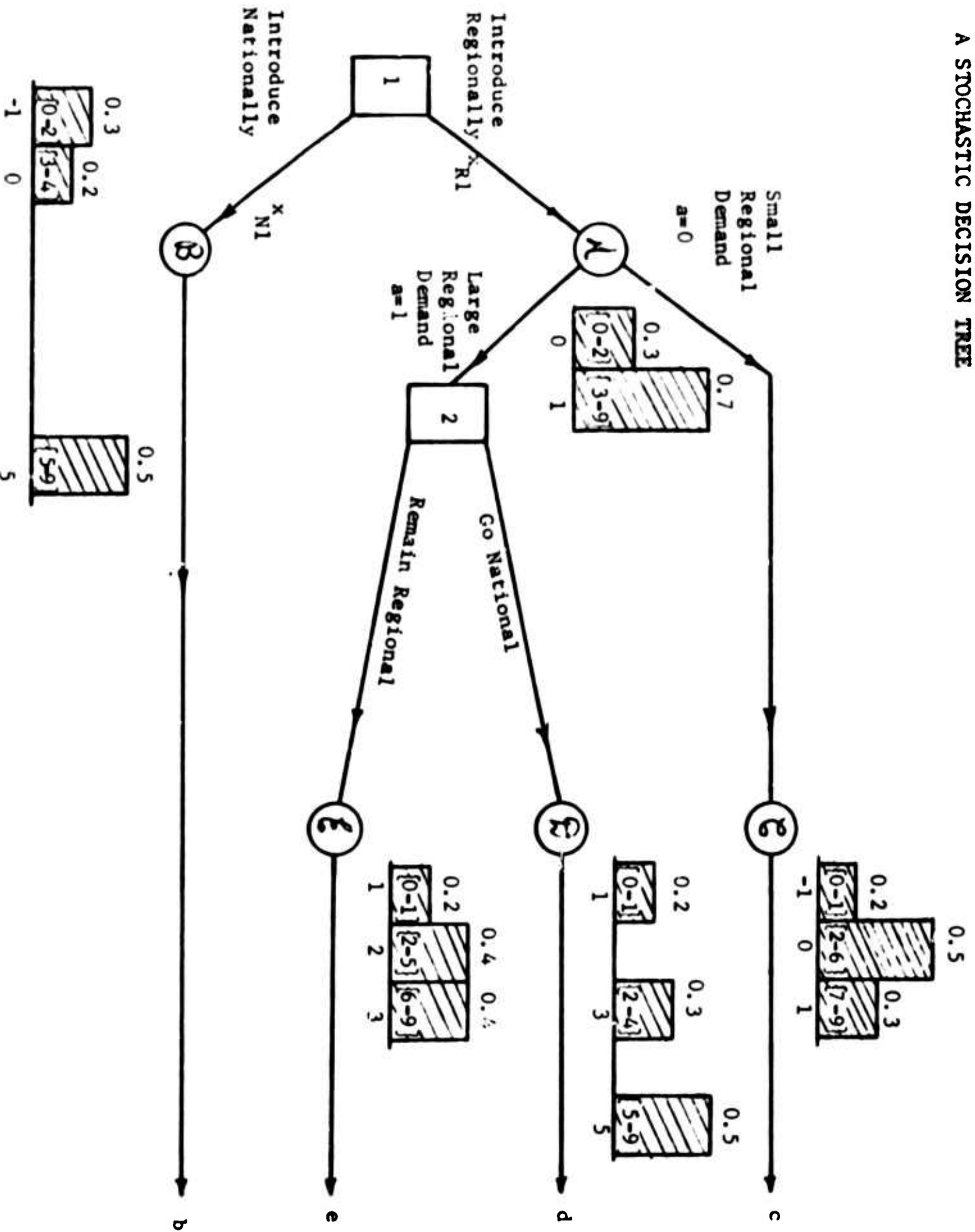
determination of whether to introduce it nationally or regionally. The
relevant data and decision alternatives are supposed to be available so
that they can be presented in a form such as Figure 1.

Technically referred to as a "tree," [1] the diagram in this Figure may be
interpreted as follows. Starting on the left a sequence of nodes and branches
may be followed to a terminus on the right. Any one of several such paths
(or routes) may be elected, but motion along any such path is always only in
the direction indicated by the arrows which are apparent on each of the
branches. The nodes are indicated by rectangles and circles. Certain
supplemental information is also displayed and will be interpreted in the
immediately following discussion.

Each rectangle is called a decision node. Each decision node represents
a place where a decision must be made by an independent decision maker. Each
branch leading away from a decision node represents one of the possible
alternative choices available to this decision maker. At the node labelled 1,
for instance, a decision may be made to introduce the product regionally or to
introduce it nationally. Suppose the former is elected. In terms of the
diagram this means that the branch leading to node $\mathscr{A}$ is followed. The latter
is a circled node and all such circled letters are called chance nodes.
A chance node represents a point at which the decision maker will discover
the response of the environment (or the state of nature). Each branch leading
away from a chance node represents the outcome of a set of chance factors.
The set of outcomes can be characterized by a probability distribution indicating

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1/ Further detailed discussion of trees and related concepts may be found in
    Chapter XIX of [8]. See also [20] and [28].

the probability that any particular outcome will occur. These probability distributions are represented on Figure 1 adjacent to each chance node. Thus here, at node $\mathcal{A}$ for instance, a play of chance factors determines whether a "Small Regional Demand" or a "Large Regional Demand" will be experienced. For the example here it is supposed that the former has a probability of occurrence of 0.3 whereas Large Regional Demand has a 0.7 probability of occurrence.

If we suppose that the latter occurs then a further movement is effected along the branch of the tree that leads into node 2. This is a rectangle. Hence at this point another decision may be made--viz., "Go National" or "Remain Regional."

Suppose the choice now is "Go National." That is, suppose this decision is elected at Node 2 which means that a prior decision to "Introduce Regionally" was made at node 1 after which a "Large Regional Demand" was experienced. Given this decision at node 2--viz., "Go National"--node $\mathcal{B}$ is then encountered via the branch for "Go National" leading out of 2. $\mathcal{B}$ is a circled node. Hence, a play of chance is again invoked. This time, however, a terminal branch is encountered. This means that the play of chance at $\mathcal{B}$ is supposed to determine an amount, d. The latter, as determined by chance, is the amount that will be received. Measured by present worth, internal rate of return, or any other suitable figure of merit, this is the amount that will emerge as a result of the sequence of decisions and chance occurrences that led to the terminus where d appears in Figure 1.

In every case the probability distributions which govern the indicated

chance selections are supposed to be known.[1] To see what this means suppose that the alternative course is taken. That is, start again at node 1 and suppose now that the decision is "Introduce Nationally". This leads to node $\mathcal{B}$. The latter is a chance node. No further decision node is encountered on this route. Hence a chance draw is now made to determine the amount b that will be obtained when this route is followed to the terminus at the bottom of Figure 1.

Chance selections are always made in terms of a known probability distribution. The ones that we shall employ are all shown in Figure 1. The probability distribution which governs the choice at $\mathcal{B}$, for instance, appears at the bottom of Figure 1 where it is to be accorded the following interpretation: There is a 0.3 probability of making a "loss"[2] of b = -1 and a 0.2 probability of breaking even at b=0. On the other hand, there is a 50-50 chance (probability = 0.5) of realizing b=5. These probabilities sum to unity. Hence all other values of b have only a zero probability of occurrence.

The histograms at nodes $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ have similar interpretations relative to the amounts c, d and e that will be obtained when these nodes are encountered. The histogram at $\mathcal{A}$, however, represents the probability distribution which governs the realization of large and small regional demand. The value 0 shown on the horizontal axis for this histogram is associated with "Small Regional Demand." The value 1 is associated with "Large Regional Demand." Thus, as already observed, the probability is 0.3 that a zero (= Small Regional

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[1] Vide Hespos and Strassmann [35] as well as Hertz [34] and Hess and Quigley [33] for further discussion of this point.

[2] As measured by any figure of merit that may be employed.

Demand) will occur and 0.7 that a one (= Large Regional Demand) will occur when node $\mathcal{A}$ is encountered.

Within the columns of each of the histograms certain digits are shown in braces. These are simply the 10 integers 0, 1, ..., 9. Members from this set of 10 integers are assigned, as indicated in each probability distribution, simply to facilitate drawing from a table of random numbers[1] in which each of these integers has an equal probability of occurrence. For instance {0-2} in the bottom histogram refers to the three integers 0, 1, 2--which constitute 0.3 of the totality of the integers 0 through 9. Similarly, {3-4} represents 0.2 of this totality while {5-9} constitutes 5 of these 10 integers, and so on. Thus, these numbers are associated with the probability of drawing, respectively, b=-1, b=0, and b=5.

In Table 1, below, we provide a specimen drawing. Here the nodes were arranged in alphabetical order and the random numbers were assigned to each of these nodes as drawn. Thus, the second random number, which is a 3, is assigned to $\mathcal{B}$. It is one of the numbers {3-4}. In this case a decision to market nationally would have yielded only b=0. (See the histogram at node $\mathcal{B}$ in Figure 1.) Since this is the case and since {3-4} are the two random numbers associated with b=0 it follows that the latter is the "b value" or result obtained on this occasion at $\mathcal{B}$. See the histogram at $\mathcal{B}$ in Figure 1 and the result shown in Table 1, below.

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[1] Vide, e.g., the definition and discussion of "random number" in E. L. Kohler, A Dictionary for Accountants [42] pp. 347-351.

Table 1

Results of a Random Draw

Node	Random No.	Result
$\mathcal{A}$	5	a = 1*
$\mathcal{B}$	3	b = 0
$\mathcal{C}$	0	c = -1
$\mathcal{D}$	8	d = 5
$\mathcal{E}$	1	e = 1

*Means "large regional demand"

Similar remarks apply to node $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$. Note, however, that the random number 5 which was drawn for $\mathcal{A}$ is associated with the occurrence of a large regional demand. Hence the value at $\mathcal{C}$ is really irrelevant and can be ignored. To put the matter differently, the node at $\mathcal{A}$ serves as a "switch"[1] which cuts in (or out) part of the diagram of Figure 1 in accordance with the instruction assigned to the number obtained in any such random drawing. Thus, in particular, the random drawing of a 5 for $\mathcal{A}$, as in Table 1, produces the situation shown in Figure 2. That is branches leading from node $\mathcal{A}$ to node $\mathcal{C}$ and beyond in Figure 1 are eliminated. The remaining random numbers--viz., those drawn for $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{E}$ --are then assigned to the remaining nodes for interpretations of the kind that have already been indicated.

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[1] Vide, e.g., pp. 650-656 in [8].

FIGURE 2

A TRIAL RESULT
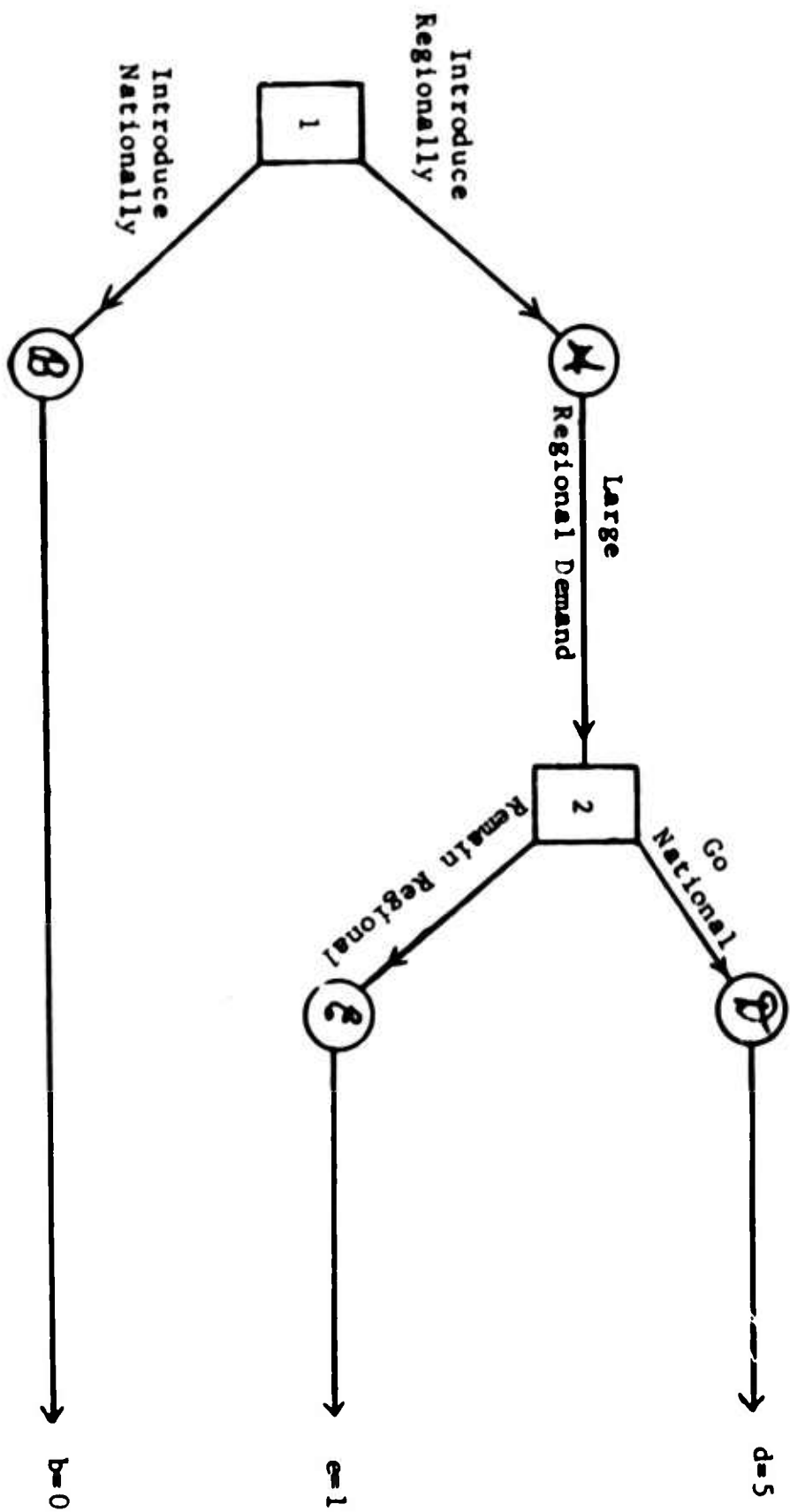
3. An Analytic Representation:

Now consider each of the following three possible strategies:

(1)

(i) Introduce nationally

(ii) Introduce regionally and if a large regional demand materializes then go national

(iii) Introduce regionally and if a large regional demand materializes then remain regional

The probability distributions of rewards and penalties for each of these strategies is wanted. In the Stochastic Decion Tree approach--as is also true in Risk and Venture Analysis--this is obtained via a series of simulation runs. In the case of Stochastic Decision Trees the rules for executing each such simulation are as follows:

(2)

(a) Each time a decision node is encountered take all branches leading out from any such node.

(b) Each time a chance node is encountered take only the one branch (or value) designed by a chance (or random) drawing which is associated with the histogram at this node.

The point of these rules is to make it possible to obtain probability distributions for all relevant combinations of decisions. Evidently for decision (i), in (1) above, one can hardly do better than simply reproduce the probability distribution for $\mathcal{B}$ in Figure 1. This is not the case for the other two decision possibilities, however, since their outcomes are influenced by the event which occurs at $\mathcal{H}$ on each trial. Thus a problem arises of combining the probability distribution at $\mathcal{H}$ with the distributions at $\mathcal{B}$, $\mathcal{A}$ and $\mathcal{E}$ in order to obtain a basis for deciding between the available alternatives.

Although the Stochastic Decision Tree approach has been described as proceeding via a simulation route, the sense of all of this can perhaps be brought together in the form of an analytical model. Here we shall employ an optimization version of such modelling possibilities because of some suggestive advantages this offers for comparison with some of the alternative possibilities that we also want to explore. Thus we write

$$\text{min.} \quad z$$

subject to

$$
\begin{aligned}
\mathcal{B}\, x_{N1} & & &= z + z_1 \\
\mathcal{C} x_S + \mathcal{D} x_L &= z &&+ z_2 \\
\mathcal{E}\, x_S + \mathcal{E}\, x_L &= z &&+ z_3 \\
\mathcal{A}\, x_{R1} &\quad - x_L &&= 0 \\
(1-\mathcal{A}) x_{R1} &\quad - x_S &&= 0 \\
x_{R1} & &&= 1 \\
x_{N1} & &&= 1
\end{aligned}
$$

(3)

where $z \geq 0$ is the scalar to be minimized.[1] I.e., our objective is "min. z" under the indicated constraints, in accordance with a two-stage procedure of the following variety. First, at stage one, the values of x_{R1} and x_{N1} are selected.[2] These values are necessarily equal to unity in the present case and hence this selection is already made. See rule (a) in

[1] This formulation is suggested by the relations for vector optimization and functional efficiency as set forth in Chapter IX of [8].

[2] See Figure 1 for their significance--viz., x_{R1} = 1 means "Introduce Regionally" and x_{N1} means "Introduce Nationally".

(2). Next, random draws are effected and the resulting values are then
assigned to the random variables $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ in these
expressions. After these data are all known, stage two is then invoked.
At this second stage the values of γ, z_1, z_2, and z_3 are determined in a
way that minimizes $\gamma \geq 0$. Evidently the minimum will always be at $\gamma = 0$.
Hence the values of z_1, z_2 and z_3 may be recorded and the results assigned,
respectively, to strategies (i), (ii) and (iii) as in (1), above, at the end
of each such trial.

For purposes of further illustration, a series of 10 random draws has
been made and the results arranged as in Table 2, below.[1] This has been
done only for nodes $\mathcal{A}$, $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ and hence only for the z_2 and z_3
values which are associated with strategies (ii) and (iii) in (1), above.
It was not necessary to effect any draws for (i) since we cannot do better
than to transfer the probability distribution assigned to node $\mathcal{B}$ in Figure 1.
The latter can therefore be compared directly with the distributions obtained
for strategies (ii) and (iii). This is accomplished by transforming the data
of Table 2 into histogram form as in Figure 3 where these histograms now serve
as approximations to the wanted probability distributions. The point here,
of course, is not the accuracy of these approximations[2] but rather the fact
that the entire probability distributions are available for considering any

[1] From page 28, line 1 of M. G. Kendall and B. Babington Smith, Tables of Random Sampling Numbers in Tracts for Computers No. XXIV, E. S. Pearson, ed. (Cambridge University Press, 1951).

[2] Many more trials would undoubtedly be necessary if the simulation route were followed as would probably be true in many practical applications.

of various dimensions of risk, penalty and reward when effecting a selection
between these alternatives. [1] Thus, in particular, the probability distribution
for strategy (i) is transferred intact from Figure 1 whereas the probability
distributions for strategies (ii) and (iii) are prepared from simulated
combinations of events at node $\mathcal{A}$ with, respectively, events at nodes
$\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$.

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[1] Additional considerations might include estimates of the probability
that the indicated strategy choices might actually be employed, etc.

Table 2

A Ten Trial Sequence of Random Number Draws

Order of Drawing*		Node				Result	
		A	C	D	E	z_2	z_3
a=1	1	4	8	1	8	1	3
	2	3	3	9	2	5	2
	5	5	9	9	3	5	2
	6	6	3	8	9	5	3
	10	4	1	5	0	5	1
a=0	3	2	6	3	1	0	0
	4	1	2	3	7	0	0
	7	1	5	8	8	0	0
	8	2	8	0	2	1	1
	9	2	5	1	8	0	0

*The order of drawing is as indicated although the data above have been regrouped into the cases a =1(Large Regional Demand) and a = 0 (Small Regional Demand).

FIGURE 3

PROBABILITY DISTRIBUTIONS
FOR
THREE STRATEGY CHOICES
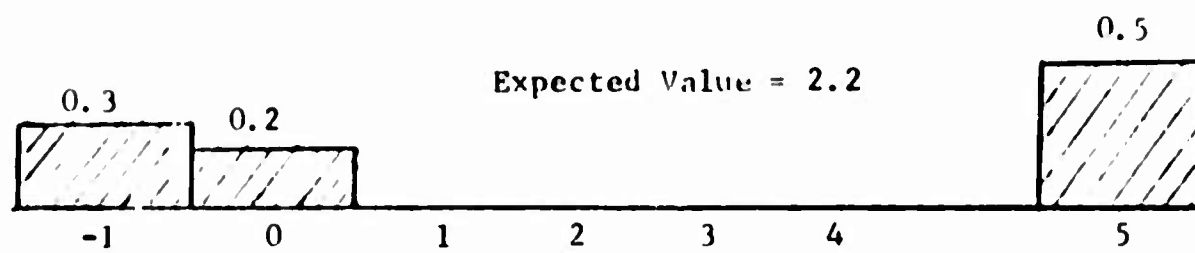
Strategy (i)

Strategy (ii)

Strategy (iii)

Note: Expected Value = Average Value as determined from the probability distribution obtained in these trials.

4.   An Approach via Stochastic Linear Programming:

Certain salient points should now be apparent.   First, no choices are really admitted among the decision variables during the simulations.   That is, chance makes all the choices in (3), by reference to the data generated in each trial.   Only after all trials have been completed is a choice then effected from the three strategies specified under (1).   The latter is made by reference to t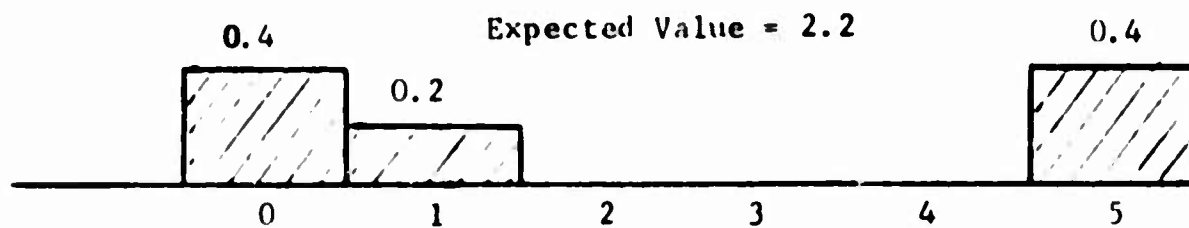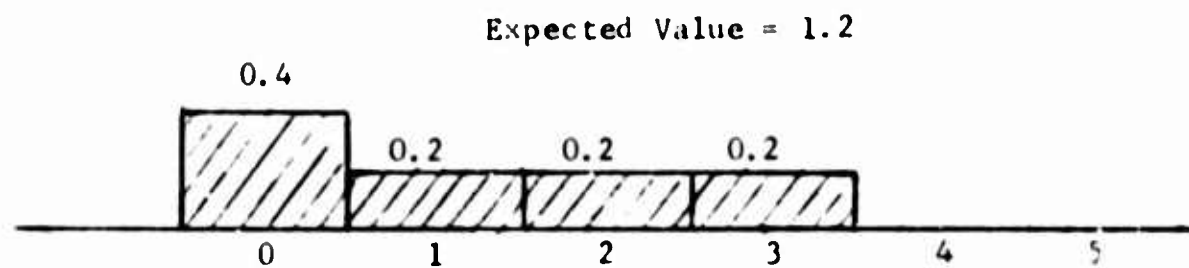he set of entire probability distributions that are then available.   Second, specific assumptions are made about the strategy possibilities relative to the adaptations that might be made over time.   The latter, in turn, also implies certain assumptions about the way the data may be expected to unfold over time as each strategy is executed, as well as how these strategies might affect the probability distributions, and so on.[1]   To put the matter differently a choice of one of the three strategy possibilities implies that a mode of implementation is decided on initially and that it remains fixed thereafter.   This carried with it certain assumptions as to data availability and so on.

An alternative to (3) may aid in sharpening some of these points.   Thus, we now refer to Figure 1 again and formulate a model as in

$$\min. \quad \gamma$$

subject to

(4)
$$\mathcal{B} x_{N1} + \mathcal{C} x_S + \mathcal{D} x_L \leq \gamma$$
$$\mathcal{B} x_{N1} + \mathcal{C} x_S + \mathcal{E} x_L \leq \gamma$$
$$\beta x_{R1} \qquad\qquad - x_L = 0$$
$$(1 - \beta) x_{R1} \qquad - x_S \qquad = 0$$
$$x_{R1} + x_{N1} \qquad\qquad = 1$$

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[1]   Cf. the discussion of the DEMON model in e.g., references [12] and   [13].

where we omit the restriction $\gamma \geq 0$ but add the requirement that x_{R1} and x_{N1} must be non-negative integers.

We now note that this last requirement means that exactly one of x_{R1}, x_{N1} will be equal to unity and the other one will be equal to zero. Thus if $x_{R1} = 1$ then the initial decision is "Introduce Regionally" while if $x_{N1} = 1$ then the initial decision is "Introduce Nationally."

We continue as before in order to treat γ as a second stage variable except that the value of "min. γ" is now chosen only after all of the chance choices have been made. This may then be interpreted as a variant of the approach called "Stochastic Linear Programming"--a name which has been accorded to yet another approach that has also been developed for dealing with decisions under risk--[1] under which (a) an initial decision maker chooses either $x_{R1} = 1$ or $x_{N1} = 1$, next (b) the random elements all materialize and then (c) a final decision maker chooses the minimizing value of γ.

For the case $x_{N1} = 1$, we evidently have to consider (as before) only the probability distribution which is already available at node $\mathcal{B}$ in Figure 1. For the case $x_{R1} = 1$ the relevant data may be obtained from Table 2[2] via the relation

(5) $$\min \ \gamma = \max \ \{z_2, \ z_3\}$$

so that, e.g., for the first draw we have min $\gamma = 3$, while for the second

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[1] Cf., e.g., Tintner [68] and [69], Tintner, Millham and Sengupta [57] and note that we are here dealing only with what is called the active case in Stochastic Linear Programming.

[2] Assuming again that these 10 trials are sufficient, at least for purposes of illustration.

draw we have min. $\gamma = z_2 = 5$, and so on. See the values for max $\{z_2, z_3\}$ in Table 2.

As contrasted with the previous model of Stochastic Decision Tree (=SDT) variety, this "min. γ" for Stochastic Linear Programming (=SLP) is evidently a random variable. As is generally the case for SLP, the probability distribution for min. γ is derived on the supposition that all relevant data will be completely determined and hence known in all detail (a) <u>after</u> the first stage decision but (b) <u>before</u> the second stage decision. Thus, on this hypothesis the choice $x_{N1} = 1$ produces a probability distribution for min. γ which we can denote as $\gamma^*(x_{N1} = 1)$. The latter is identical with the first distribution given in Figure 3. But the case $\gamma^*(x_{R1} = 1)$ does not produce a probability distribution which coincides with any of those shown in Figure 3. Instead, via (5), we obtain the one shown in Figure 4 which can supposedly be compared with the one for Strategy (i) in Figure 3 when trying to decide whether to choose $x_{N1} = 1$ or $x_{R1} = 1$ at stage 1.

FIGURE 4

PROBABILITY DISTRIBUTION

for $\gamma^*(x_{R1} = 1)$
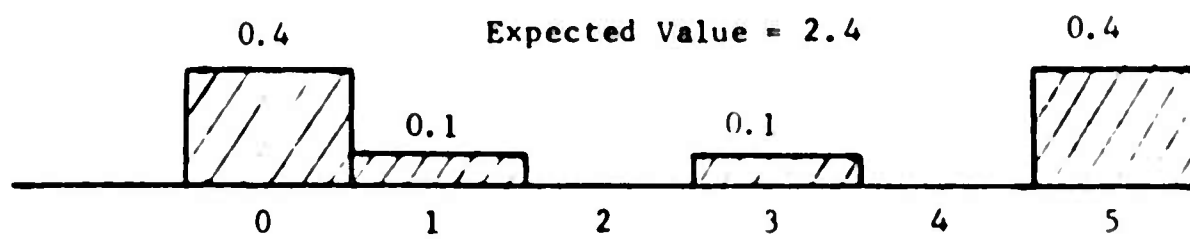
5.   Expected Value Choices and Risk Control Constraints:

Each of the above approaches evidently imposes different assumptions about the state of information at decision node 2 in Figure 1. Hence a different route of adaptation is pursued in each case and the compound probability distributions which should be considered at stage 1 (relative to stage 2) also differ.   E.g., Stochastic Linear Programming (=SLP) assumes that all of the relevant data will be known with certainty when (or if) node 2 is reached. In adaptations to this state of information, SLP naturally directs attention to the probability distribution of max (z_2, z_3) = $\gamma^*(x_{R1} = 1)$ for comparison with the distribution of z_1, the latter being the alternative first-stage choice.

The Stochastic Decision Tree (= SDP) approach assumes that the position in the tree is always known along with (a) the remaining alternatives and (b) the relevant probability distributions. It should be emphasized, however, that the stage 1 choice under SDT represents a complete commitment. Thus if the probability distribution for (ii) in (1) were to have the most appeal, then the strategy associated with this selection would be followed unconditionally-- viz , the decision "Go National" would always be made whenever node 2 was encountered after an initial decision to distribute regionally. This is not the case for SLP, however, under the same initial decision. That is, given an initial decision to distribute regionally, the information-adaptation assumptions for SLP would make these stage 2 choices contingent on the outcomes [1] of the random events.

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[1]   In the "passive case" of SLP even these stage 1 choices need not be made in advance. See Tintner et. al. [57] and [68].

In order to obtain further perspective on each of the above approaches, it is helpful to turn to still other approaches that are now available for dealing with the problem of choice under risk. As was noted at the outset of this paper, some of these approaches proceed via the optimization of a single figure of merit and one such approach is available via the expected value optimizations (under constraints) which are characterized as "Linear Programming Under Uncertainty."[1]

This, too, may be illustrated by reference to Figure 1 and in a way that will help to relate it to the preceding and well as subsequent examples and topics. Thus we therefore write our LPUU (= Linear Programming Under Uncertainty) model in order to achieve a very simple illustration as follows:

$$\min E \left[c_{N1} x_{N1} + c_{R1} x_{R1} + \gamma + c_2 z_2 + c_3 z_3 \right]$$

subject to

$$
\begin{aligned}
\mathcal{B} x_{N1} &+ \mathcal{D} x_L + \mathcal{C} x_S = \gamma - z_2 \\
\mathcal{B} x_{N1} &+ \mathcal{E} x_L + \mathcal{C} x_S = \gamma \quad - z_3 \\
(1 - \mathcal{A}) x_{R1} &- x_S = 0 \\
\mathcal{A} x_{R1} - x_L &= 0 \\
x_{N1} + x_{R1} &= 1
\end{aligned}
$$

(6)

where "E" refers to "expected value" in the functional to be minimized. As before, x_{N1} and x_{R1} are first-stage variables which are required to be non-negative. The variables γ, z_2 and z_3 are second-stage choices with z_2, $z_3 \geq 0$ but γ otherwise unconstrained. The other symbols in (6) have

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[1] The term is due to G. B. Dantzig. See [21] and [22].

all been explained before except for c_{N1}, c_{R1}, c_2 and c_3 which may be interpreted as benefit and penalty rates per unit use of the variables with which they are associated.

As was the case for SLP, we again assume that the second stage choices are made only _after_ the data are available--e.g., after each drawing has been made as in, say, Table 1 and 2. In fact, setting $c_{N1} = c_{R1} = c_2 = c_3 = 0$ the objective becomes min $E\gamma$ and so in this case the preceding distributions for $\gamma(x_{R1} = 1)$ and $\gamma(x_{N1} = 1)$ become relevant. In principle, the model of (6) sets forth all relevant data, including the probability distributions, in explicit detail. The choice, however, is by reference to the expected value minimization only. Of course, there is no reason why the relevant probability distributions cannot also be presented especially when (as in the present case) they are already available.

For ease of reference we present the distribution for $\gamma^*(x_{R1} = 1)$ and $\gamma^*(x_{N1} = 1)$ in Table 3. The latter is transferred directly from Figure 1, but we do **not** transfer the former from Table 2 where the results of our 10 trials are given. Instead we utilize exact relations. That is, in Table 3 we use $Pr(\gamma^* | \mathcal{A} x_{R1} = 1)$ to signify "the probability of securing the indicated value of $\gamma^* (= \min \gamma)$, given that $\mathcal{A} x_{R1} = 1$." Similarly, $Pr(\gamma^* | (1 - \mathcal{A}) x_{R1} = 1)$ [1] refers to the probability of the only other possibility when $x_{R1} = 1$. Then we compute the relevant probabilities via

(7) $Pr(\gamma^* = z | \mathcal{A} x_{R1} = 1) Pr(\mathcal{A} x_{R1} = 1) + Pr(\gamma^* = z | (1-\mathcal{A}) x_{R1} = 1)$

   $Pr((1 - \mathcal{A}) x_{R1} = 1) = Pr(\gamma^* = z)$

---

[1]   Naturally, both of these probabilities are _identically_ zero when $x_{R1} = 0$.

The latter result is entered in column 4 of Table 4 using $\Pr(\mathcal{A} x_{R1} = 1) = 0.7$ and $\Pr((1 - \mathcal{A}) x_{R1} = 1) = 0.3$. The distribution for γ^* when $x_{N1} = 1$ is then shown for comparison in the final column of this table.

Table 3

Probability Distributions
for
$$\gamma^* = \max \{z_2, z_3\} \text{ and } \gamma^* = z_1$$

		$\gamma^*(x_{R1} = 1)$			$\gamma^*(x_{N1} = 1)$
	Column No.			Probability Distribution	Probability Distribution
(1)	(2)	(3)		0.7x column (2) plus 0.3x column (3)	From Figure 1 Node $\mathcal{B}$
γ^*	$\Pr(\gamma^* \mid \mathcal{A} \, x_{R1} = 1)$	$\Pr(\gamma^* \mid (1 - \mathcal{A}) x_{R1} = 1)$			
-1	0.00	0.20		0.060	0.300
0	0.00	0.50		0.150	0.200
1	0.04	0.30		0.118	0.000
2	0.08	0.00		0.056	0.000
3	0.38	0.00		0.266	0.000
4	0.00	0.00		0.000	0.000
5	0.50	0.00		0.350	0.500
Total	1.00	1.00		1.000	1.000
Expected Value	3.84	0.10		2.718	2.200

Referring to Table 3 we observe that $E_\gamma(x_{R1} = 1) = 2.718$ while $E_\gamma(x_{N1} = 1) = 2.2$ and hence we might prefer the former to the latter on this expected value criterion. [1]/ But what about other aspects of these distributions? These would also seem to require consideration from the stand-point of possible realizations of other γ^* values.

To handle these additional aspects of the problem one might proceed to alter the constants in the functional of (6). Specifically one might alter these constants to provide penalties and incentives for the different γ^* values and their associated probabilities of realization. This would, in fact, be the normal procedure for dealing with such risks under LPUU. [2]/

Another alternative would involve introducing further aspects of risk control and allowance by inserting additional constraints. There is no reason to suppose that this must preclude any recourse to adjustment of the functional constants. That is, both of these approaches can be used simultaneously in LPUU and, of course, the approach of LPUU can also be joined to still other approaches. [3]/

To close this section we now illustrate one such additional possibility, and for this we might introduce a constraint of the Chance Constrained Programming ($=C^2$) variety by writing

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1/   This would implicitly change the objective of to

$$\max_{x_{N1}, x_{R1}} \quad \min_{\gamma} \quad E \gamma$$

where $\gamma = \max \{z_2, z_3\}$ or z_1.

2/   Vide, e.g., Dempster [24].

3/   See, e.g., [7] which represents a first instance in which ideas of LPUU were joined to those of Chance Constrained Programming. Although the illustration in [7] proceeds by reference to capital budgeting under liquidity and payback constraints this is not the only possibility. The crucial development which opened the way to this is in [16] which will also admit of other uses as well.

(8)
$$Pr(\gamma \geq z) \geq \alpha.$$

Here z is a prescribed value of γ, called a quality level, and $0 \leq \alpha \leq 1$ measures the risks (= $1 - \alpha$) that this level will not be met. That is the choice of x and the subsequent minimization of γ must yield a $\gamma^* < z$ at most $(1 - \alpha)$ proportion of the time. Thus, as this interpretation is meant to suggest, (8) is to be regaided as an additional constraint--i.e., a Chance Constraint--which is to be considered in conjunction with all of the other constraints and the functional in (6).

Note that we are now exploring the alternative approach to risk that is available via C^2 Programming but in a way that also relates it to the preceding developments. Thus, suppose, for instance, that the quality level is set at $z = 5.0$ and the risk level at $\alpha = 0.5$. This would block the probability distribution for $\gamma^*(x_{R1} = 1)$ from consideration since the probability is only 0.35 that a value of $\gamma^* \geq z = 5.0$ or more will be achieved. The first stage choice will then be $x_{N1} = 1$. But something more than only this decision possibility is also now available. For instance, certain risk evaluations and sensitivity analyses can be executed. E.g., in expected value terms, an opportunity cost can be imputed to this degree of avoidance of risk at the indicated quality level in the amount $0.518 = 2.718 - 2.200$. (See the expected values listed in the bottom row of the last two columns in Table 3.) Furthermore, if the risk protection value exceeded $\alpha = 0.5$ at this quality level (viz., $z = 5.0$) then there would be no solution. That is, neither $x_{N1} = 1$ nor $x_{R1} = 1$ would be acceptable and then either the decision to market this new product would need to be reversed or else a revision of the risk-control

constraint would be in order.[1] Note that it is by no means certain that
the latter represents the only possibility.[2] For instance in actual
practice[3] it may be supposed that unless a value of $z = 5.0$ is actually
attained then it is really not feasibl. to try to market this product since
retail outlets will not be willing to handle the product at
lower volumes. In such cases this constraint would then provide a measure
of risk of infeasibility or inapplicability of the model for the situations
to which it is supposed to apply. Alternatively, it might also be used to
control additional dimensions of opportunity cost. It might be the case
for example that a firm would not wish to market this product and thereby
tie up executive and other talents unless it can achieve at least the indicated
quality level, z, with at least the indicated probability, α.

6.   Information Assumptions and Decision Rules:

The introduction of chance constraints, as in (8), immediately
raises the issue of decision rules--what they are, how they might be
characterized or prescribed, and how they might be selected. This is so
because their employments have been a built-in feature of C^2 programming
from its inception.[4]

1/   See, A. Charnes and W. W. Cooper [10] for a development relative to
      aspiration level theory in social psychology and the "satisficing"
      models and constructs of H. A. Simon.

2/   That is, it is by no means certain that these aspirations would be
      revised only because a decision to market the product in one way or
      another had already been made.

3/   Vide, e.g., the discussions in [12] and [13].

4/   See, e.g., [15] as well as the earlier references cited there.

A full-scale development of this topic would require introducing possibilities for altering the probability distributions and collecting information on such alteration possibilities.[1] A very simple illustration will suffice,[2] however, and to this end the structural aspects of the model are written in the following form:

$$\mathcal{B}\,x_{N1} \qquad\qquad +\mathcal{D}\,x_{N2} + \mathcal{E}\,x_{R2} + \mathcal{C}\,x_S \;=\; \gamma$$

(9.1)
$$(1-\mathcal{A})x_{R1} \qquad\qquad -\; x_S \;=\; 0$$

$$\mathcal{A}\,x_{R1} \;-\; x_{N2} \;-\; x_{R2} \qquad\qquad =\; 0$$

$$x_{N1} \;+\; x_{R1} \qquad\qquad\qquad =\; 1$$

The chance constraint

(9.2)
$$\Pr[\gamma \geq z] \geq \alpha$$

is also adjoined to (9.1). Finally the objective is formulated as:

(9.3)
$$\max\; E\,\gamma$$

As before x_{N1} and x_{R1} represent first-stage decision variables which are constrained to be non-negative integers. The same non-negative integer requirement is also imposed on x_{N2} and x_{R2}. The latter pair may be regarded as a further decomposition of the variable x_L which appeared in all of the preceding models. These two variables--viz., x_{N2} and x_{R2}--are now regarded as

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[1] Cf., e.g., any of the referneces to DEMON type models where these aspects of C^2 programming are developed.

[2] Vide also Hespos and Strassmann [35] who make this point as a way of joining the decision tree approach of Magee [47] and the risk analysis of Hertz [34].

second stage decision variables that are introduced to represent the respective alternatives of "Go National" and "Remain Regional" when node 2 is encountered in Figure 1.

Using the same conventions as previously, the value $x_{N2} = 1$ means "Go National" and $x_{R2} = 1$ means "Remain Regional." The requirement $\mathcal{A} x_{R1} = x_{N2} + x_{R2}$ in (9.1) means that unless $x_{R1} = 1$ at stage 1 there cannot be a second stage choice since if $x_{R1} = 0$ then, necessarily $\mathcal{A} x_{R1} = 0$. On the other hand, the choice $x_{R1} = 1$ does not suffice to produce $\mathcal{A} x_{R1} = 1$. This is to say that the possibility of a second stage choice depends on a random event--viz., the event "Large Regional Demand" at Node $\mathcal{A}$ in Figure 1. This second stage choice must evidently then be delayed until knowlege of the value of $\mathcal{A} x_{R1}$ is at hand. Given $\mathcal{A} x_{R1} = 1$, however, then a second-stage decision _must_ be made[1] and one of the two available choices taken out of node 2. I.e., either $x_{N2} = 1$ or $x_{R2} = 1$ when $\mathcal{A} x_{R1} = 1$.

The possibilities implicit in the preceding remarks are summarized somewhat more succinctly by writing

(10)
$$x_{N2} = \max \{0, \mathcal{A} x_{R1}\}$$
or
$$x_{R2} = \max \{0, \mathcal{A} x_{R1}\}$$

---

[1] We are interpreting this in accordance with the spirit of the Hespos-Strassman analysis [35]. Other interpretations would allow cases such as $\mathcal{A} x_{R1} \geq x_{N2} + x_{R2}$, etc.

which can then be interpreted as the relevant possibilities for a decision rule to complete a first-stage choice of $x_{R1} = 1$. These can then be compared with each other and with the decision rule $x_{N1} = 1$ in order to determine which is the best rule to follow.

A decision rule must cover every contingency that may be confronted. That is, it must specify the choice to be made at every decision node. This in turn implies that the data needed to implement the decision rule must be at hand. Hence provided the data on $\mathscr{A} x_{R1}$ are at hand by the time node 2 is encountered, the above rules are unambiguous.

The rules indicated under (10) evidently conform to the way Figure 1 was interpreted and so do the choices $x_{R1} = 1$ and $x_{N1} = 1$. Furthermore, the rules must satisfy the constraints and this is evidently the case for (9.1).

Refer now to the data in Table 4 below, however, and suppose that (9.2) prescribes $\alpha = 0.5$ and $z = 3$. Evidently the rule $x_{R1} = 1$, $x_{R2} = \max \{0, \mathscr{A} x_{R1}\}$ does not satisfy this constraint since the column of probabilities for this rule gives $\Pr\{\mathfrak{z} \geq \mathfrak{z}\} = 0.28$ which is less than the requisite $\alpha = 0.5$. Hence this rule is eliminated and the competition reduces to $x_{N1} = 1$ vs $x_{R1} = 1$, $x_{N2} = \max \{0, \mathscr{A} x_{R1}\}$. Both rules satisfy all constraints in (9) but the latter has a higher expected value. Hence the latter rule is best among all of the feasible decision rules.

Table 4

Probability Distributions for Two Decision Rules
when $X_{R1} = 1$

VALUE	Stage	Decision Rule	
	1	$x_{R1} = 1$	
	2	$x_{N2} = \max \{0, \quad x_{R1}\}$	$x_{R2} = \max \{0, \quad x_{R1}\}$
-1		0.06	0.06
0		0.15	0.15
1		0.23	0.23
2		0.00	0.28
3		0.21	0.28
4		0.00	0.00
5		0.35	0.00
Total		1.00	1.00
Expected Value		2.55	1.57

7. <u>Conclusion and Summary</u>:

Unlike the earlier situation for SLP it is now supposed that only a knowledge of the relevant probability distributions is available at each decision node. In any event an optimal rule has now been designated. Hence it might be supposed that the model should be discarded. This is true insofar as the problem is regarded as "solved" once the indicated rule is at hand. On the other hand, it may be prudent to return to further uses of the model. These might include "sensitivity analyses" under which the given probability distributions (e.g., as in Figure 1) are altered in order to study what variations can occur in choices between the decision rules that can be elected. They might also include extended dual evaluator analyses $\underline{1/}$ and so on.

The model might itself also be altered in the light of these findings. For instance, in the last section even the initial decision was reversed so that in this case the Chance Constraints served as a feasibility check on the previous decision assumptions relative to company risk control policies and known market volume level requirements.

The illustrations used in this paper were all synthesized by reference to examples in which y served as the only measure for the figure of merit in both the objective and the constraints. This served to simplify the examples but, of course, it in no way implies that either risks or these new approaches are limited in this manner. Other additional constraints (chance or otherwise) might also be employed for physical as well as financial dimensions of a problem.

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$\underline{1/}$ See, e.g., [52].

Multi-dimensional aspects might also include both bogey (implicit) rates of return and payback as well as present worth all in a single comprehensive model which also included   still other dimensions for controlling and evaluating risk-vs.-opportunity possibilities. As a case in point we might employ a Chance Constraint on cash expenditure rates in order to ensure against risk of insolvency. Then we might also introduce one or more constraints on payback [1] as a hedge against uncertainty. For instance we may not be sure whether even better opportunities may be available at some future date. Then, to hedge against this uncertainty, we might specify a C^2 approach to payback in which an initial investment is to be recovered by a specified time (payback period) with a prescribed level of probability. This might then be evaluated by altering the stipulated payback period (quality level) as well as the stipulated probability (risk level) of achieving it. These opportunity cost consequences being available in the form of suitable measures of expected profit, say, it should then be possible to array these on at least an ordinal probability scale as a basis for matching and evaluating such uncertainty possibilities.

We have discussed the latter kinds of topics in other writings where we have also (a) suggested ways of extending present payback practices and (b) shown how LPUU and C^2 programming can be joined together for these (and other) purposes. [2]   By devices like these it is possible, in any event, to

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[1]   E.g., stipulating that certain cumulative proportions of an initial investment must be returned by prescribed times at specified levels of probability. See, e.g., [7] for discussions of this and other extensions to ordinary usages of payback.

[2]   See [7].

handle multi-dimensional aspects of risk such as may be present in many
management problems. After a sufficient experience over a variety of such
problems with any particular management it might then be possible to synthesize
some variant of a utility function approach. That is, it might then be
possible to synthesize a function that would make the same choices that this
management might make over a wide variety of alternatives. Here we leave
aside such issues as the measurement of satisfactions, etc., since their emphasis
is not really pertinent to the present context. The point to emphasize is,
rather, that this accomplishment is terminal rather than an initial act for the
methods that we are discussing.[1]

In conclusion we now turn to the tree concept itself. This concept, as we
have elsewhere shown [11],[2] can be related to the processes of double-entry
accounting and hence to related aspects of budgeting and financial planning.
Conversely, these same double-entry accounting processes can be related to
the corresponding tree and network concepts. But this is meant to imply that
the latter approach, along with the related analytical models, may be a better
way of utilizing the double-entry principle--at least when probability distri-
butions are to be compounded for such purposes as (a) investment selection
and (b) projection of profit-and-loss statement categories along with the related
balance-sheet and flow-of-funds analyses. This might be done analytically,
of course, as well as by computer simulation or by a combination of these means

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[1] Tintner [69] does refer to a utility function synthesized ab initio as
     a basis of choice, but it does not play a central role in his initial or
     subsequent developments.

[2] See also [8] Chapter XVII, XIX and XX, as well as Ijiri [40].

as circumstances may warrant. The resulting statements of flow or
position might then be reported in terms of entire probability distributions
or by single-number (best) estimates as circumstances and managerial
convenience may warrant. The approaches covered in this paper suggest some
of the possibilities that might be employed to this end either singly or in
combination. Note, in conclusion, that this would then involve elaborations
of the tree concept so that multiple probability distributions might then be
required at any node or branch along with specified rules and relations for effecting
their combinations in terms of their interlocking relations to each other
and to the relevant dimensions of risk and profit.

APPENDIX

Because LPUU and C^2 Programming are descendents of ordinary linear programming it may be useful to consider the latter topic, too, insofar as this can be done by reference to the problem of chosing an optimum decision rule. For this purpose we might first proceed by reference to Chance Constraints on risk formulated as follows. Thus suppose, for instance, the constraint relative to the distribution at $\mathscr{B}$ is formulated as

$$(11.1) \qquad x_{N1} \sum_{b=z}^{5} Pr(b) \geq \alpha \, x_{N1}$$

which means that the quality level z is to be maintained with risk no greater than $1-\alpha$ of attaining z or more.

Given that $0 \leq \alpha \leq 1$ is a stipulated constant we may see what this means by supposing that (11.1) refers to the probability of breaking even. That is, we are supposing, say, that a level z=0 or greater is to be attained with probability α. Reference to Table 3 (or Figure 1) would then provide the relevant data for substitution in (3) as in the following expression

$$(11.2) \qquad x_{N1} \sum_{b=0}^{5} Pr(b) = x_{N1}(0.2 + 0.5) \geq \alpha \, x_{N1}$$

Now we observe that this constraint can always be satisfied by choosing $x_{N1} = 0$ but a value of $x_{N1} = 1$ can be designated only if $\alpha \leq 0.7$.

Evidently different quality levels might be imposed for the risks that might attend different paths through the tree of Figure 1. Therefore proceeding somewhat more abstractly than in (11.2) we now write the remaining risk constraints in the form

$$x_{N2} \sum_{d=z}^{5} \Pr(d \mid \mathcal{A}=1) \geq x_{N2} (\alpha + \delta) - x_{R1} \delta$$

(12.1)

$$x_{R2} \sum_{e=z}^{5} \Pr(e \mid \mathcal{A}=1) \geq x_{R2} (\alpha + \delta) - x_{RL} \delta$$

and also

$$x_{R1} \sum_{c=z}^{5} \Pr(c \mid \mathcal{A}=0) \Pr(\mathcal{A}=0) + x_{N2} \sum_{d=z}^{5} \Pr(d \mid \mathcal{A}=1) \Pr(\mathcal{A}=1) +$$

(12.2)

$$+ x_{R2} \Pr(e \mid \mathcal{A}=1) \Pr(\mathcal{A}=1) \geq x_{R1} (\alpha + \delta) - x_{N2} \delta - x_{R2} \delta$$

where the latter, along with (12.1) and (11.1) are to be simultaneously satisfied with

$$x_{N1} + x_{R1} = 1$$

(13)

$$x_{N2} + x_{R2} = \mathcal{A} x_{P1} \leq 1 .$$

As before all decision variables (which have already been explained) must be integer valued. The symbol δ, which is new, refers to a constant which is "sufficiently large"[1] so that in the first constraint of (12.1), say, a choice of $x_{N2} = 1$ could not be made unless also $x_{R1} = 1$. (E.g., any choice of δ so that $\alpha + \delta > 1$ would suffice.) Because $\delta > 0$ and the left-hand member of each expression in (12.1) is non-negative, the choice $x_{R1} = 1$ is not precluded in any case. On the other hand, reference to (12.2)

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[1] See, e.g., the discussion of regularization techniques in linear programming as discussed in Chapter VIII of [8].

shows that $x_{R1} = 1$ cannot be designated unless at least one of the choices $x_{N2} = 1$ or $x_{R2} = 1$ is also made. Finally, the conditions in (13), along with the restriction to integer choices, means that at most one of the expressions $x_{N2} = 1$ or $x_{R2} = 1$ can hold.

We now observe that the constraint sets in (11) and (12) might first be scrutinized to see whether any of the a priori choice possibilities are to be deleted because the related linear inequalities can be fulfilled only with choices of the decision variables at x=0. Supposing that this eventuality does not emerge we may then proceed with reference to the objective function [1] which in this case may be formulated as follows:

(14)
$$\max \ y = E \ \beta \ x_{N1} + E(\mathcal{E}|\mathcal{A}=0) \ Pr(\mathcal{A}=0) \ x_{R1}$$
$$+ [E(\mathcal{K}|\mathcal{A}=1) \ x_{N2} + E(\mathcal{E}|\mathcal{A}=1)x_{R2}] \ Pr(\mathcal{A}=1)$$

The above symbols may best be explained by writing their numerical values as

$$E(\mathcal{E}|\mathcal{A}=1) \ Pr(\mathcal{A}=1) = (2.2) \ (0.7) = 1.54$$
$$E(\lambda|\mathcal{A}=1) \ Pr(\mathcal{A}=1) = (3.6) \ (0.7) = 2.52$$
$$E(\mathcal{E}|\mathcal{A}=0) \ Pr(\mathcal{A}=0) = (0.1) \ (0.3) = 0.03$$
$$E(\beta) \qquad\qquad = 2.20$$

where the indicated computations are made directly from the probability

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[1] Vide Ehrenfeld and Littauer [25] pp.117 ff.

distributions given in Figure 1. In short $E(\mathcal{E}|\mathcal{A}=1)$ refers to the probability distribution shown at node $\mathcal{E}$ in Figure 1 while the values of $E(\mathcal{D}|\mathcal{A}=1)$ and $E(\mathcal{E}|\mathcal{A}=0)$ are obtained from, respectively, the distributions shown at nodes $\mathcal{D}$ and $\mathcal{E}$.

In this case then the functional in (14) may also be represented as

$$(15) \qquad \max. \; \mathcal{Y} = 2.20x_{N1} + 0.03x_{R1} + 2.52x_{N2} + 1.54x_{R2}$$

with the indicated maximization to be undertaken with respect to the constraining relations (11), (12) and (13). Save for the second constraint in (13), this would be an ordinary linear programming problem. The condition

$$x_{N2} + x_{R2} - \mathcal{A} \; x_{R1} \leq 1$$

means that we cannot set either x_{N2} or $x_{R2} = 1$ until after $\mathcal{A}$ has materialized. On the other hand, we can choose one of these variables to be zero in advance of any knowledge of the specific a that will materialize. Reference to (15) makes it clear that this zero value should be assigned to x_{R2} in the present case and then the further choice should be $x_{R1} = 1$ so that the optimal rule is, as noted in the text, $x_{R1} = 1$, $x_{N.} = \max \; \{0, \mathcal{A} \; x_{R1}\}$.

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     Relatively recent innovations in methods for risk analysis are here
surveyed and related by means of certain linear programming characterizations
applied to venture and risk analysis, stochastic decision trees, stochastic
linear programming, linear programming under uncertainty, and chance
constrained programming. Possibilities for combining these approaches in
various ways are also discussed and illustrated by example. Implications are
noted for accounting, budgeting and other aspects of management planning.

14		LINK A		LINK B		LINK C	
KEY WORDS		**ROLE**	**WT**	**ROLE**	**WT**	**ROLE**	**WT**
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